Erasure and Termination in Higher-Order Rewriting

Jeroen Ketema and Femke van Raamsdonk

Faculty of Sciences, Vrije Universiteit, Amsterdam, The Netherlands
jketema@cs.vu.nl
femke@cs.vu.nl

Abstract. Two applications of the Erasure Lemma for first-order orthogonal term rewriting systems are: weak innermost termination implies termination, and weak normalization implies strong normalization for non-erasing systems. We discuss these two results in the setting of higher-order rewriting.

1 Introduction

The Erasure Lemma for orthogonal term rewriting systems (TRSs) states that if we lose the possibility of performing an infinite reduction by reducing s to t, then a subterm that admits an infinite reduction is erased. Two well-known results can be seen as applications of the Erasure Lemma. The first, due to O'Donnell [8], states that weak innermost normalization implies termination for orthogonal TRSs. We give an example showing that O'Donnell's result cannot directly be generalized to the higher-order case, and we propose a restricted version for the higher-order setting. The second result originates from Church, and states that weak and strong normalization coincide for orthogonal and non-erasing TRSs. Klop extends this to second-order rewriting. We discuss the definitions of non-erasing and propose one that permits to extend Church's result to (general) higher-order rewriting.

2 Notation

We work with higher-order rewriting systems (HRSs) as defined by Nipkow [7]; we also refer to combinatory reduction systems (CRSs) defined by Klop [4].

A term s is *terminating* or *strongly normalizing*, notation $\mathsf{SN}(s)$, if all rewrite sequences starting in s are finite. It is *weakly normalizing*, notation $\mathsf{WN}(s)$, if there is a (finite) sequence $s \to^* n$ from s to a normal form n. We use the notation $\infty(s)$ to indicate that the term s admits an infinite rewrite sequence.

A rewrite step $s \to t$ is *critical* if $\infty(s)$ but not $\infty(t)$. A rewrite step $s \to t$ that is not critical is *perpetual*; in that case we have $\infty(s) \Rightarrow \infty(t)$. A rewrite step $s \to t$ is *innermost*, denoted $s \to_i t$, if no proper subterm of the contracted redex is itself a redex. A term s is *weakly innermost normalizing*, notation $\mathsf{WIN}(s)$, if there is an innermost reduction $s \to_i^* n$ to a normal form n. It is *strongly innermost normalizing*, notation $\mathsf{SIN}(s)$, if all innermost reductions starting in s are finite.

A term s has the *conservation property* if every step from s is perpetual. A term s is *uniformly normalizing* if $\mathsf{WN}(s) \Rightarrow \mathsf{SN}(s)$.

A set of terms satisfies a property if all terms in the set satisfy that property. We then also use the notations as above without the term argument. For instance, we write that a TRS is SN. As remarked in [6], uniform normalization implies the conservation property, but the reverse implication only holds for sets of terms that are closed under reduction. For instance, in the ARS $\{a \to a, a \to b, b \to b, b \to c\}$ the set $\{a\}$ has the conservation property because every step from a is perpetual, but $\{a\}$ is not uniformly normalizing.

3 O'Donnell

The Erasure Lemma (Proposition 4.8.4 and Lemma 9.3.27 in [10]) for first-order orthogonal TRSs states that in a critical step a sub-term u with $\infty(u)$ is erased. O'Donnell [8] shows that for orthogonal TRSs we have $\mathsf{WIN} \Rightarrow \mathsf{SN}$. This can be shown using the Erasure Lemma. The following example shows that the result by O'Donnell cannot directly be generalized to the higher-order case.

Example 1. Consider the HRS defined by the following rewrite rules:

$$\begin{array}{ll} f(X) & \to a \\ g(\lambda x.F(x)) & \to F(g(\lambda x.f(x))) \end{array}$$

This is a second-order orthogonal fully extended HRS. It is not SN because we have the following infinite reduction:

$$g(\lambda x.\, f(x)) \to f(g(\lambda x.\, f(x))) \to f(f(g(\lambda x.\, f(x)))) \to \ldots$$

However, the term $g(\lambda x.\, f(x))$ has an innermost reduction to normal form:

$$g(\lambda x.\, f(x)) \to_i g(\lambda x.\, a) \to_i a$$

The HRS is WIN; it is even SIN. This can be shown by giving a measure on terms that strictly decreases with each innermost reduction step. Hence the implications $\mathsf{WIN} \Rightarrow \mathsf{SN}$ and $\mathsf{SIN} \Rightarrow \mathsf{SN}$ do not hold for second-order orthogonal fully extended HRSs.

One might wonder where the proof of $\mathsf{WIN} \Rightarrow \mathsf{SN}$ breaks down in the higher-order case. First we consider the proof using the Erasure Lemma. The problem here is that the Erasure Lemma does not hold for higher-order rewriting. Indeed, the Erasure Lemma for second-order rewriting, given in [3], states the following: In a critical step a subterm u is erased, that can descend, by reduction steps that do not overlap with u, to a subterm u' with $\infty(u')$. In the example, the critical step $g(\lambda x.\, f(x)) \to g(\lambda x.\, a)$ erases the subterm x which itself does not admit an infinite reduction. However, its descendant $g(\lambda x.\, f(x))$ does; note that it is obtained by contracting the g-redex that does not overlap with x.

An alternative proof of $\mathsf{WIN} \Rightarrow \mathsf{SN}$ is given in [9], by showing the sequence of implications $\mathsf{WIN} \Rightarrow \mathsf{SIN} \Rightarrow \mathsf{SN}$ for non-overlapping TRSs. The generalization of O'Donnell's result to the case of non-overlapping instead of orthogonal TRSs is due to Gramlich [1,2]. The proof given in [9] uses besides [1,2] also [5]. The implication $\mathsf{WIN} \Rightarrow \mathsf{SIN}$ holds for non-overlapping HRSs; the proof directly carries over from the first-order case. However, Example 1 shows that the implication $\mathsf{SIN} \Rightarrow \mathsf{SN}$ does not hold for second-order orthogonal fully-extended HRSs. The proof of $\mathsf{SIN} \Rightarrow \mathsf{SN}$ for the first-order case uses the following: if in a reduction step $s \to t$ a redex is contracted that is not convergent (i.e. both confluent and SN), then $\phi(s) \to^+ \phi(t)$. Here ϕ computes the result of reducing all maximal convergent subterms to their (unique) normal form. This is Lemma 5.6.6 in [9]. Example 1 shows that this statement is not true for orthogonal second-order fully extended HRSs: In the step $g(\lambda x.\, f(x)) \to f(g(\lambda x.\, f(x)))$ the non-terminating g-redex is contracted. If we compute in both terms the normal forms of the maximal convergent subterms, then we find $g(\lambda x.\, a)$ and $f(g(\lambda x.\, a))$. There is no reduction sequence consisting of one or more steps from $g(\lambda x.\, a)$ to $f(g(\lambda x.\, a))$.

Is it possible to impose additional restrictions such that the implication $\mathsf{WIN} \Rightarrow \mathsf{SN}$ holds for the higher-order case? One possibility is to restrict attention to *bounded* orthogonal fully extended HRSs. Using the generalization of [10, Lemma 9.2.28] to the higher-order case, which states that for orthogonal fully extended HRSs, WN implies acyclicity, we conclude SN from the fact that the length of the reducts of all terms are bounded. Another possibility is to restrict attention to non-erasing rules which actually brings us to the second application of the Erasure Lemma discussed in the next section.

To conclude, note that for orthogonal TRSs the (stronger) local version of O'Donnell's result, the statement $\forall t.\,[\mathsf{WIN}(t) \Rightarrow \mathsf{SN}(t)]$, also follows from the Erasure Lemma. Klop [4, Remark 5.9.8.1(ii)] remarks that the local version of O'Donnell's result does not hold for orthogonal CRSs. For instance, the λ-term $(\lambda x.(\lambda y.z)(xx))(\lambda u.uu)$ is SIN but not SN. (This is a solution to [10, Exercise 4.8.13].)

4 Church

Church proved that weak and strong normalization coincide in the λI-calculus, where in an abstraction $\lambda x.\,M$ there is at least one free occurrence of x in M. The crucial property that makes the λI-calculus uniformly normalizing is the fact that it is *non-erasing*. That is, intuitively, a reduction step cannot erase a subterm.

A first-order rewrite rule $l \to r$ is said to be non-erasing if all variables in l also occur in r. A rewrite step is non-erasing if the applied rule is non-erasing. Orthogonal and non-erasing TRSs are uniformly normalizing (see [10, Theorem 4.8.5] and [9, Theorem 5.6.10(2)]).

For second-order rewriting, this requirement on the rewrite rule does not guarantee that the induced step is non-erasing. Clearly the step $(\lambda x.\,y)\,z \to y$ is erasing although in the β-reduction rule all variables in the left-hand side also occur in the right-hand side. Klop [4] therefore defines a CRS (which is a second-order rewriting system) to be non-erasing if for every rewrite step $s \to t$ the terms s and t have the same free variables. He then proves uniform normalization for orthogonal and non-erasing CRSs. This generalizes the result by Church because the λI-calculus is an example of an orthogonal and non-erasing CRS.

For third-order rewriting, this second-order notion of non-erasing is not sufficient for proving uniform normalization, as shown in the following example.

Example 2. Consider the HRS defined by the following rules:

$$\begin{aligned} f(\lambda x.\,F(x)) &\to F(f(\lambda x.\,F(x))) \\ g(\lambda y.\,G(y)) &\to G(\lambda u.\,a) \end{aligned}$$

This is a third-order orthogonal fully extended HRS. (In fact for the left-hand side of the g-rule we should write $g(\lambda y.\,G(\lambda z.\,y(z)))$.) Consider the term $g(\lambda y.\,f(\lambda x.\,y(x)))$. It is a redex with respect to the g-rule using the substitution $\{G \mapsto \lambda u.\,f(\lambda x.\,u(x))\}$. It contains a redex with respect to the f-rule using the substitution $\{F \mapsto \lambda u.y(u)\}$. If first the g-redex is contracted, we obtain a reduction to normal form:

$$g(\lambda y.\,f(\lambda x.\,y(x))) \to f(\lambda x.\,a) \to a$$

Note that both steps are non-erasing in the sense of Klop because all free variables (none) are preserved. But repeatedly contracting the f-redexes yields an infinite rewrite sequence:

$$g(\lambda y.\,f(\lambda x.\,y(x))) \to g(\lambda y.\,y(f(\lambda x.\,y(x)))) \to \ldots$$

Hence in a third-order orthogonal fully extended HRS a non-erasing step is not necessarily perpetual. This is already remarked in [3] where another example is given.

The question is how to adapt the definition of non-erasure such that it is useful for proving uniform normalization. Note that a rewrite step in a HRS is obtained in two stages:

$$l\sigma\downarrow_\beta \longrightarrow_R r\sigma \longrightarrow_\beta^* r\sigma\downarrow_\beta$$

Intuitively, a rewrite step is non-erasing if both stages are non-erasing.

In a first-order HRS, which is in fact a TRS, the second stage, the β-reduction to normal form, does not play a role. In that case, indeed the requirement that the applied rewrite rule is non-erasing guarantees that the rewrite step is non-erasing.

In a second-order HRS, the second stage may play a role. There is no erasure in the β-reduction to normal form if this is a λI-reduction. Observe that all abstractions in the second stage originate from the substitution σ. This motivates the following alternative to the second-order definition of non-erasure due to Klop: a rewrite step in a second-order HRS is non-erasing if all variables in the left-hand side of the applied rule also occur in the right-hand side, and in addition the substitution used to instantiate the rewrite rule is non-erasing in the sense that the terms assigned to free variables are λI-terms. This is a more restrictive definition than the one by Klop: the step in the example $f(\lambda x.\, a) \to a$ does not remove free variables, but it uses the erasing substitution $\{F \mapsto \lambda x.a\}$.

In a third-order HRS, the abstractions that occur in the second stage may originate from the substitution σ, but also from the arguments of the free variables in the rewrite rule. In the example, the free variable G in the g-rule has as argument the abstraction $\lambda u.\, a$. Note that this is not a λI-term. In a second-order HRS, free variables cannot have abstractions as argument. The second stage is a λI-reduction if both the abstractions originating from σ and the abstractions originating from the rewrite rule are λI-terms.

To summarize, a rewrite step in a HRS is non-erasing if first, all variables in the left-hand side of the applied rewrite rule also occur in the right-hand side, and second, the applied substitution assigns λI-terms to free variables, and third, the arguments of the free variables in the applied rewrite rule are λI-terms. We claim that for orthogonal HRSs weak and strong normalization coincide if all steps are non-erasing, and that non-erasing steps are perpetual in orthogonal fully extended HRSs.

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